

# A REFINED LUECKING'S THEOREM AND FINITE-RANK PRODUCTS OF TOEPLITZ OPERATORS

TRIEU LE

**ABSTRACT.** For any function  $f$  in  $L^\infty(\mathbb{D})$ , let  $T_f$  denote the corresponding Toeplitz operator the Bergman space  $A^2(\mathbb{D})$ . A recent result of D. Luecking shows that if  $T_f$  has finite rank then  $f$  must be the zero function. Using a refined version of this result, we show that if all except possibly one of the functions  $f_1, \dots, f_m$  are radial and  $T_{f_1} \cdots T_{f_m}$  has finite rank, then one of these functions must be zero.

## 1. INTRODUCTION

As usual, let  $\mathbb{D}$  denote the unit disk and  $\mathbb{T}$  denote the unit circle in the complex plane  $\mathbb{C}$ . Let  $dA$  denote the Lebesgue measure on  $\mathbb{D}$  which is normalized such that the unit disk has total mass 1. We have  $dA(z) = \frac{1}{\pi} dx dy$ , where  $z = x + iy$  for  $x, y$  real. We write  $L^2$  for  $L^2(\mathbb{D}, dA)$ . The Bergman space  $A^2$  is the subspace of  $L^2$  that consists of holomorphic functions. It is well-known that  $A^2$  is a closed subspace of  $L^2$ . The standard orthonormal basis for  $A^2$  is  $\{e_m : m = 0, 1, \dots\}$ , where  $e_m(z) = \sqrt{m+1} z^m$  for any non-negative integer  $m$ . Let  $P$  denote the orthogonal projection from  $L^2$  onto  $A^2$ . For any function  $f \in L^2$ , the Toeplitz operator with symbol  $f$  is denoted by  $T_f$ , which is densely defined on  $A^2$  by  $T_f \varphi = P(f\varphi)$  for  $\varphi \in H^\infty$  - the space of all bounded holomorphic functions on  $\mathbb{D}$ . The operator  $T_f$  is in fact an integral operator by the formula

$$(T_f \varphi)(z) = \int_{\mathbb{D}} \frac{f(w) \varphi(w)}{(1 - \bar{w}z)^2} dA(w), \text{ for } z \in \mathbb{D}, \varphi \in H^\infty.$$

If  $f$  is a bounded function then  $T_f$  is a bounded operator on  $A^2$  with  $\|T_f\| \leq \|f\|_\infty$  and  $(T_f)^* = T_{\bar{f}}$ . However, unbounded symbol can also give rise to bounded Toeplitz operators. In fact, since  $T_f$  is an integral operator with kernel  $\frac{f(w)}{(1 - \bar{w}z)^2}$  for  $z, w \in \mathbb{D}$ , we see that if  $f \in L^2$  supported in a compact subset of  $\mathbb{D}$  then  $T_f$  is a compact operator on  $A^2$ .

---

2000 *Mathematics Subject Classification.* Primary 47B35.

*Key words and phrases.* Toeplitz operator, Bergman space, finite-rank product.

A function  $f$  on  $\mathbb{D}$  is called a radial function if we have  $f(z) = f(|z|)$  for almost all  $z \in \mathbb{D}$ . If  $f \in L^2$  is radial then using polar coordinate we see that

$$\begin{aligned} \langle T_f e_m, e_k \rangle &= \sqrt{(m+1)(k+1)} \int_{\mathbb{D}} f(z) z^m \bar{z}^k dA(z) \\ &= \begin{cases} 0 & \text{if } m \neq k \\ (m+1) \int_0^1 2f(t) t^{2m+1} dt & \text{if } m = k \end{cases} \\ &= \begin{cases} 0 & \text{if } m \neq k \\ (m+1) \int_0^1 f(r^{1/2}) r^m dr & \text{if } m = k. \end{cases} \end{aligned}$$

This shows that the operator  $T_f$  is diagonal with respect to the standard orthonormal basis. The eigenvalues of  $T_f$  are given by

$$\omega(f, m) = \langle T_f e_m, e_m \rangle = (m+1) \int_0^1 f(r^{1/2}) r^m dr, \quad m = 0, 1, \dots \quad (1.1)$$

It follows from Stone-Weierstrass's Theorem that if  $f \in L^2$  such that  $T_f$  is the zero operator then  $f$  must vanish almost everywhere in  $\mathbb{D}$ . On the other hand, the problem of determining whether there exists a nontrivial finite rank Toeplitz operator on  $A^2$  was open for quite a long time. Recently D. Luecking has found an elegant proof that gives the negative answer to this problem.

There is an extensive literature on Toeplitz operators on the Hardy space  $H^2$  of the unit circle. We refer the reader to [9] for definitions of  $H^2$  and their Toeplitz operators. In the context of Toeplitz operators on  $H^2$ , it was showed by A. Brown and P. Halmos [3] back in the sixties that if  $f$  and  $g$  are bounded functions on the unit circle then  $T_g T_f$  is another Toeplitz operator if and only if either  $f$  or  $\bar{g}$  is holomorphic. From this it is readily deduced that if  $f, g \in L^\infty(\mathbb{T})$  such that  $T_g T_f = 0$  then one of the symbols must be the zero function. In contrast with this, for Toeplitz operators on the Bergman space, it has not been known if it is true that for  $f, g \in L^\infty(\mathbb{D})$ ,  $T_g T_f = 0$  implies  $g$  or  $f$  is the zero function. Affirmative answers have been obtained by researchers only in special cases. In [1], P. Ahern and Ž. Čučković answered this problem affirmatively with the assumption that both  $f$  and  $g$  are bounded harmonic functions on  $\mathbb{D}$ . Later in [4], Čučković was able to show that if  $f, g$  are bounded such that  $f$  is harmonic and  $g(re^{i\theta}) = \sum_{m=-\infty}^N g_m(r) e^{im\theta}$  for  $z = re^{i\theta} \in \mathbb{D}$ , then  $T_g T_f = 0$  implies either  $f = 0$  or  $g = 0$ . The case one of the symbols is a bounded radial function has also been understood. See [2] and [7] for more details. In fact, in [7], the author was able to show that if all except possibly one of the functions  $f_1, \dots, f_M$  are bounded radial functions and  $T_{f_1} \cdots T_{f_M} = 0$  then one of these functions must be zero.

A more general problem than the above zero product problem is the finite rank product problem. Recall that the above mentioned theorem of Luecking shows that if  $f \in L^2$  such that  $T_f$  has finite rank then  $f$  is the zero

function. What happens if  $T_g T_f$  has finite rank, where  $f$  and  $g$  are bounded measurable functions on the unit disk? The answer to this general question seems to be still far from completed but the following important case has been understood: If  $f$  and  $g$  are bounded harmonic functions then one of them must be the zero function (K. Guo, S. Sun and D. Zheng [6]). The purpose of this paper is to report the same answer in some other special cases.

In the first part of this paper, we use Luecking's Theorem to show that if  $f, g$  are functions in  $L^2$  where  $f$  satisfies a certain condition and  $T_g T_f$  (which is densely defined on  $A^2$ ) has finite rank, then either  $f = 0$  or  $g = 0$ . In the second part of the paper, we prove a "refined" version of Luecking's Theorem and use it to show that if  $f_1, \dots, f_{m_1}$  and  $g_1, \dots, g_{m_2}$  are radial functions in  $L^\infty$  and  $f$  is a function in  $L^2$  such that  $T_{g_1} \cdots T_{g_{m_2}} T_f T_{f_1} \cdots T_{f_{m_1}}$  (which is densely defined on  $A^2$ ) has finite rank, then one of the above functions must be zero.

## 2. FINITE RANK PRODUCTS OF TWO TOEPLITZ OPERATORS

We begin this section with a detailed discussion of the decomposition  $L^2 = \bigoplus_{m \in \mathbb{Z}} \mathcal{R} e^{im\theta}$ , where

$$\mathcal{R} = \{u : [0, 1) \rightarrow \mathbb{C} \text{ such that } \int_0^1 |u(r)|^2 r dr < \infty\}.$$

This decomposition has been used by Čučković and Rao in their studies of Toeplitz operators (see Section 2 in [5]). Let  $f \in L^2(\mathbb{D})$ . Then for almost all  $r \in [0, 1)$ , the function  $\zeta \mapsto f(r\zeta)$  for  $\zeta \in \mathbb{T}$  is in  $L^2(\mathbb{T}, \frac{1}{2\pi} d\theta)$ . Since  $\{\zeta^m : m \in \mathbb{Z}\}$  is an orthonormal basis for  $L^2(\mathbb{T}, \frac{1}{2\pi} d\theta)$ , we have

$$f(r\zeta) = \sum_{m=-\infty}^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-im\theta} d\theta \right) \zeta^m,$$

where the sum takes place in  $L^2(\mathbb{T})$ . For  $m \in \mathbb{Z}$ , define

$$f_m(r) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{-im\theta}) d\theta, \quad 0 \leq r < 1.$$

Then the above representation becomes (with  $\zeta = e^{i\theta}$ ),

$$f(re^{i\theta}) = \sum_{m=-\infty}^{\infty} f_m(r) e^{im\theta}. \quad (2.1)$$

This representation holds for almost all  $r \in [0, 1)$  and for such  $r$ , the sum on the right hand side takes place in  $L^2(\mathbb{T})$ . Now we have

$$\begin{aligned} \|f\|_{L^2(\mathbb{D})}^2 &= \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right) r dr \\ &= \int_0^1 \left( \sum_{m=-\infty}^{\infty} |f_m(r)|^2 \right) r dr \\ &= \sum_{m=-\infty}^{\infty} \int_0^1 |f_m(r)|^2 r dr. \end{aligned}$$

This shows that  $f_m \in \mathcal{R}$  for all  $m \in \mathbb{Z}$  and the right hand side of (2.1) converges in  $L^2(\mathbb{D})$ . Therefore the representation (2.1) in fact takes place in  $L^2(\mathbb{D})$ .

The following theorem is our first result in the paper.

**Theorem 2.1.** *Suppose  $f \in L^2$  with  $f(re^{i\theta}) = \sum_{m=-\infty}^M f_m(r)e^{im\theta}$  for  $z = re^{i\theta}$ ,*

*where  $M$  is an integer. Assume that  $\int_0^1 f_M(r)r^k dr \neq 0$  for all  $k \geq N$ , where  $N$  is a positive integer. If  $g \in L^2$  such that  $T_g T_f$  (which is densely defined on  $A^2$ ) has finite rank then  $g$  is the zero function.*

*Proof.* Recall that  $A^2(\mathbb{D})$  has the orthonormal basis  $\{e_m : m = 0, 1, \dots\}$ , where  $e_m(z) = \sqrt{m+1} z^m$  for any non-negative integer  $m$ . For any non-negative integers  $k, l$  we have

$$\begin{aligned} \langle T_f e_k, e_l \rangle &= \sqrt{(k+1)(l+1)} \int_{\mathbb{D}} f(z) z^k \bar{z}^l dA(z) \\ &= \sqrt{(k+1)(l+1)} \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{i(k-l)\theta} d\theta \right) r^{k+l+1} dr \\ &= \sqrt{(k+1)(l+1)} \int_0^1 f_{l-k}(r) r^{k+l+1} dr. \end{aligned}$$

By assumption about  $f$ ,  $\langle T_f e_k, e_l \rangle = 0$  whenever  $l - k > M$ . Thus for  $k \in \mathbb{N}$  such that  $k + M \geq 0$ , we have

$$\begin{aligned} T_f e_k &= \sum_{l=0}^{\infty} \langle T_f e_k, e_l \rangle e_l \\ &= \sqrt{k+1} \sum_{l=0}^{k+M} \left( \sqrt{l+1} \int_0^1 f_{l-k}(r) r^{k+l+1} dr \right) e_l \end{aligned}$$

$$\begin{aligned}
&= \sqrt{(k+1)(M+k+1)} \left( \int_0^1 f_M(r) r^{2k+M+1} dr \right) e_{k+M} \\
&\quad + \sqrt{k+1} \sum_{l=0}^{k+M-1} \left( \sqrt{l+1} \int_0^1 f_{l-k}(r) r^{k+l+1} dr \right) e_l
\end{aligned}$$

This shows that when  $k+M \geq 1$  and  $2k+M+1 \geq N$ ,  $e_{k+M}$  can be written as a linear combination of  $\{T_f e_k\} \cup \{e_0, \dots, e_{k+M-1}\}$ .

Now suppose  $T_g T_f$  has finite rank and let  $\{\varphi_1, \dots, \varphi_K\}$  is a set that spans  $T_g T_f(\mathcal{P})$  where  $\mathcal{P}$  is the space of all polynomials in the variable  $z$ . Then for any non-negative integer  $k$  with  $k+M \geq 1$  and  $2k+M+1 \geq N$  we see that  $T_g e_{k+M}$  is a linear combination of  $\{\varphi_1, \dots, \varphi_K\} \cup \{T_g(e_0), \dots, T_g(e_{k+M-1})\}$ . From this, it follows by induction that  $T_g$  is a finite rank operator. By Luecking's Theorem [8] or a refined version of it (see Theorem 3.1 in Section 3), we see that  $g$  is the zero function.  $\square$

**Remark 2.2.** If  $f(z) = \bar{h}(z) + p(z, \bar{z})$  where  $h \in A^2$  and  $p$  a polynomial in two variables then  $f$  can be written in the form in the hypothesis of Theorem 2.1. Therefore, Theorem 2.1 shows that if  $T_g T_f$  is of finite rank for some  $g \in L^2$  then either  $f$  or  $g$  is the zero function.

### 3. A REFINED LUECKING'S THEOREM AND FINITE RANK PRODUCTS OF TOEPLITZ OPERATORS

We begin this section by a refined version of Luecking's Theorem whose proof is greatly influenced by Luecking's argument. For the rest of the paper, let  $\mathcal{P}$  denote the space of all polynomials in the variable  $z$ .

**Theorem 3.1.** Let  $\mathcal{S} \subset \mathbb{N}$  ( $\mathbb{N}$  denotes the set of all non-negative integers) so that  $\sum_{s \in \mathcal{S}} \frac{1}{s+1} < \infty$ . Let  $\mathcal{N}$  be the subspace of  $\mathcal{P}$  spanned by the monomials  $\{z^m : m \in \mathbb{N} \setminus \mathcal{S}\}$  and let  $\mathcal{N}^* = \{\bar{g} : g \in \mathcal{N}\}$ . Let  $\nu$  be a complex regular Borel measure on  $\mathbb{C}$  with compact support. Let  $T_\nu$  be the operator from  $\mathcal{N}$  to the space of linear functionals on  $\mathcal{N}^*$  by  $T_\nu f(\bar{g}) = \int_{\mathbb{C}} f \bar{g} d\nu$  for all  $f, g \in \mathcal{N}$ . Then  $T_\nu$  has finite rank if and only if the support of  $\nu$  is finite.

*Proof.* For any  $z \in \mathbb{C}$ , let  $\delta_z$  denote the point mass measure concentrated at  $z$ . Since  $T_{\nu - \nu(\{0\})\delta_0} = T_\nu - \nu(\{0\})T_{\delta_0}$ , we see that  $T_\nu$  has finite rank if and only if  $T_{\nu - \nu(\{0\})\delta_0}$  has finite rank. So without loss of generality, we may assume that  $\nu(\{0\}) = 0$ .

If the support of  $\nu$  is contained in a finite set  $\{z_1, \dots, z_{N-1}\}$  for some  $N \geq 2$ , then  $T_\nu = \sum_{j=1}^{N-1} \nu(\{z_j\})T_{\delta_{z_j}}$ . Hence  $T_\nu$  has rank less than  $N$ .

Conversely, suppose  $T_\nu$  has rank less than  $N$ . Following Luecking's argument in [8, p. 3], we see that for any  $f_1, \dots, f_N$  and  $g_1, \dots, g_N$  in  $\mathcal{N}$ ,

$$\int_{\mathbb{C}^n} \prod_{l=1}^N f_l(z_l) \det(\bar{g}_i(z_j)) d\nu^N(Z) = 0, \tag{3.1}$$

where  $Z = (z_1, \dots, z_N) \in \mathbb{C}^N$  and  $\nu^N$  is the product of  $N$  copies of  $\mu$  on  $\mathbb{C}^N$ .

Let  $m_1, \dots, m_N$  and  $k_1, \dots, k_N$  be non-negative integers. Let

$$\begin{aligned} \mathcal{Z} &= \{s \in \mathbb{N} : s + m_j \notin \mathcal{S} \text{ and } s + k_j \notin \mathcal{S} \text{ for all } 1 \leq j \leq N\} \\ &= \mathbb{N} \setminus \left( \left( \bigcup_{j=1}^N (\mathcal{S} - m_j) \right) \cup \left( \bigcup_{j=1}^N (\mathcal{S} - k_j) \right) \right). \end{aligned}$$

Since  $\sum_{s \in \mathcal{S}} \frac{1}{s+1} < \infty$  we have  $\sum_{s \in \mathbb{N} \setminus \mathcal{Z}} \frac{1}{s+1} < \infty$ . This shows that

$$\sum_{s \in \mathcal{Z}} \frac{1}{s+1} = \infty. \quad (3.2)$$

Now for any  $s \in \mathcal{Z}$ , the monomials  $f_j(z) = z^{m_j+s}$  and  $g_j(z) = z^{k_j+s}$  for  $j = 1, \dots, N$  are not in  $\mathcal{N}$ . So we may use (3.1) to get

$$\begin{aligned} 0 &= \int_{\mathbb{C}^n} \prod_{l=1}^N z_l^{m_l+s} \det(\bar{z}_j^{k_i+s}) d\nu^N(Z) \\ &= \int_{\mathbb{C}^n} \prod_{l=1}^N z_l^{m_l} \det(\bar{z}_j^{k_i}) |z_1 \dots z_N|^{2s} d\nu^N(Z) \\ &= \int_{\mathbb{C}^n \setminus W} \prod_{l=1}^N z_l^{m_l} \det(\bar{z}_j^{k_i}) |z_1 \dots z_N|^{2s} d\nu^N(Z), \end{aligned} \quad (3.3)$$

where  $W = \{Z = (z_1, \dots, z_N) \in \mathbb{C}^N : z_1 \dots z_N = 0\}$ . The last identity follows from the fact that  $\nu^N(W) = 0$ .

Let  $\mathbb{K}$  denote the open right half plane consisting of all  $w$  with  $\Re(w) > 0$  and let  $\bar{\mathbb{K}}$  denote the closure of  $\mathbb{K}$  in  $\mathbb{C}$ . For any  $w \in \mathbb{K}$ , define

$$F(w) = \int_{\mathbb{C}^n \setminus W} \prod_{l=1}^N z_l^{m_l} \det(\bar{z}_j^{k_i}) |z_1 \dots z_N|^{2w} d\nu^N(Z).$$

Here, for a positive number  $t$  and a complex number  $w$ ,  $t^w = \exp(w \log t)$  where  $\log$  is the principal branch of the logarithmic function.

Suppose the measure  $\nu$  is supported in the disk  $D(0, R)$  of radius  $R > 0$  centered at the origin in the complex plane. Then  $\nu^N$  is supported in the polydisk  $D_N(0, R)$  of the same radius centered at the origin in  $\mathbb{C}^N$ . Then for any  $w \in \mathbb{K}$  and any  $Z = (z_1, \dots, z_N)$  in the above polydisk, we have

$$|z_1 \dots z_N|^{2w} = |z_1 \dots z_N|^{2\Re(w)} \leq R^{2N\Re(w)}.$$

Therefore,

$$|F(w)| = \left| \int_{D_N(0, R) \setminus W} \prod_{l=1}^N z_l^{m_l} \det(\bar{z}_j^{k_i}) |z_1 \dots z_N|^{2w} d\nu^N(Z) \right| \leq C R^{2N\Re(w)},$$

where  $C$  is a constant independent of  $w$ . It follows that  $F$  is not only defined on  $\bar{\mathbb{K}}$  but also continuous on  $\bar{\mathbb{K}}$ . Now an application of Morera's Theorem

shows that  $F$  is analytic on  $\mathbb{K}$ . Let  $G(w) = F(w)R^{-2Nw}$  for  $w \in \mathbb{K}$ , then  $G$  is continuous, bounded on  $\bar{\mathbb{K}}$  and analytic on  $\mathbb{K}$ . Now define

$$H(\zeta) = G\left(\frac{1+\zeta}{1-\zeta}\right) \quad (|\zeta| < 1).$$

Then  $H$  is a bounded analytic function on the unit disk. For any  $s \in \mathcal{Z}$ , equation (3.3) and the definitions of  $F, G$  show that  $G(s) = F(s) = 0$ , which implies  $H\left(\frac{s-1}{s+1}\right) = 0$ . Now

$$\sum_{\substack{s \in \mathcal{Z} \\ s \geq 1}} \left(1 - \left|\frac{s-1}{s+1}\right|\right) = \sum_{\substack{s \in \mathcal{Z} \\ s \geq 1}} \frac{2}{s+1} = \infty \quad \text{by (3.2).}$$

Corollary to Theorem 15.23 in [10] shows that  $H$  is identically zero on the unit disk. Hence  $G$  and  $F$  are identically zero in  $\bar{\mathbb{K}}$ . In particular,  $F(0) = 0$ , which shows that

$$\int_{\mathbb{C}^n} \prod_{l=1}^N z_l^{m_l} \det(\bar{z}_j^{k_i}) d\nu^N(Z) = 0.$$

Since  $m_1, \dots, m_N$  and  $k_1, \dots, k_N$  were arbitrary non-negative integers, we conclude that (3.1) holds for all  $f_1, \dots, f_N$  and  $g_1, \dots, g_N$  in  $\mathcal{P}$ . Following Luecking's argument again [8, Section 4 and 5], we conclude that the support of  $\nu$  is finite.  $\square$

Now let  $\mathcal{S}$  and  $\mathcal{N}$  be as in the hypothesis of Theorem 3.1. Let  $\mathcal{M}$  denote the subspace of  $\mathcal{P}$  spanned by  $\{z^m : m \in \mathcal{S}\}$ . Let  $\bar{\mathcal{M}}$  (respectively,  $\bar{\mathcal{N}}$ ) denote the closure of  $\mathcal{M}$  (respectively,  $\mathcal{N}$ ) in  $A^2$ .

**Corollary 3.2.** *Suppose  $f \in L^2$  so the operator  $T_f$  is densely defined on  $A^2$ . If  $T_f(\mathcal{N}) \subset \text{Span}(\bar{\mathcal{M}} \cup \{\varphi_1, \dots, \varphi_N\})$ , where  $\varphi_1, \dots, \varphi_N \in A^2$ , then  $f$  is the zero function.*

*Proof.* Let  $P_{\bar{\mathcal{M}}}$  (respectively,  $P_{\bar{\mathcal{N}}}$ ) denote the orthogonal projection from  $A^2$  onto  $\bar{\mathcal{M}}$  (respectively,  $\bar{\mathcal{N}}$ ). Then we have  $P_{\bar{\mathcal{N}}} = 1 - P_{\bar{\mathcal{M}}}$  and hence  $P_{\bar{\mathcal{M}}}P_{\bar{\mathcal{N}}} = P_{\bar{\mathcal{N}}}P_{\bar{\mathcal{M}}} = 0$ . By replacing  $\varphi_j$  by  $\varphi_j - P_{\bar{\mathcal{M}}}\varphi_j$  if necessary, we may assume that  $\varphi_j \perp \mathcal{M}$  for  $1 \leq j \leq N$ . By using the Gram-Schmidt process if necessary, we may assume that the vectors  $\varphi_1, \dots, \varphi_N$  are orthonormal (we may have fewer vectors after using Gram-Schmidt process but let us still denote by  $N$  the total number of the vectors).

For any  $p$  in  $\mathcal{N}$  we have  $T_f p = P_{\bar{\mathcal{M}}}T_f p + \sum_{j=1}^N \langle T_f p, \varphi_j \rangle \varphi_j$ . This shows that  $P_{\bar{\mathcal{N}}}(T_f p) = \sum_{j=1}^N \langle T_f p, \varphi_j \rangle P_{\bar{\mathcal{N}}}\varphi_j = \sum_{j=1}^N \langle T_f p, \varphi_j \rangle \varphi_j = \sum_{j=1}^N \langle f p, \varphi_j \rangle \varphi_j$ . Then for any  $q$  in  $\mathcal{N}$ , we have

$$\int_{\mathbb{D}} f p \bar{q} \, dA = \langle T_f p, q \rangle = \langle P_{\bar{\mathcal{N}}}(T_f p), q \rangle = \sum_{j=1}^N \langle f p, \varphi_j \rangle \langle \varphi_j, q \rangle.$$

Let  $d\nu = f dA$ . Then the map  $T_\nu$  from  $\mathcal{N}$  to the space of linear functionals on  $\mathcal{N}^*$  defined by  $T_\nu p(\bar{q}) = \int_{\mathbb{D}} p\bar{q} d\nu = \int_{\mathbb{D}} f p\bar{q} dA$  for  $p, q \in \mathcal{N}$  is of finite rank. Now Theorem 3.1 shows that the support of  $\nu$  is finite, which implies that  $f(z) = 0$  for almost all  $z \in \mathbb{D}$ .  $\square$

**Theorem 3.3.** *Suppose  $f_1, \dots, f_{m_1}$  and  $g_1, \dots, g_{m_2}$  are radial functions in  $L^\infty$  none of which is the zero function. Suppose  $f$  is a function in  $L^2$  such that  $T_{g_1} \cdots T_{g_{m_2}} T_f T_{f_1} \cdots T_{f_{m_1}}$  (which is densely defined on  $A^2$ ) is of finite rank, then  $f$  must be the zero function.*

*Proof.* For any  $h \in \{f_1, \dots, f_{m_1}, g_1, \dots, g_{m_2}\}$ , the operator  $T_h$  is diagonal with eigenvalues  $\omega(h, m)$  given by (1.1) for  $m = 0, 1, \dots$ . Let  $Z(h) = \{m \in \mathbb{N} : \omega(h, m) = 0\}$ . Since  $h$  is not the zero function, Müntz-Szász's Theorem (see [10, Theorem 15.26]) shows that  $\sum_{m \in Z(h)} \frac{1}{m+1} < \infty$ .

Let  $\mathcal{S} = Z(f_1) \cup \cdots \cup Z(f_{m_1}) \cup Z(g_1) \cup \cdots \cup Z(g_{m_2})$ . Then we have  $\sum_{m \in \mathcal{S}} \frac{1}{s+1} < \infty$ . Let  $\mathcal{N}$  (respectively,  $\mathcal{M}$ ) is the subspace of  $\mathcal{P}$  spanned by  $\{e_m : m \in \mathbb{N} \setminus \mathcal{S}\}$  (respectively,  $\{e_m : m \in \mathcal{S}\}$ ). Recall that  $\mathcal{P}$  denotes the space of all analytic polynomials in the variable  $z$ .

Put  $S_1 = T_{f_1} \cdots T_{f_{m_1}}$  and  $S_2 = T_{g_1} \cdots T_{g_{m_2}}$ . For  $\varphi \in A^2$  we have

$$S_2 \varphi = T_{g_1} \cdots T_{g_{m_2}} \left( \sum_{j=1}^{\infty} \langle \varphi, e_j \rangle e_j \right) = \sum_{j=1}^{\infty} \omega(g_1, j) \cdots \omega(g_{m_2}, j) \langle \varphi, e_j \rangle e_j.$$

Hence if  $S_2 \varphi = 0$ , then  $\omega(g_1, j) \cdots \omega(g_{m_2}, j) \langle \varphi, e_j \rangle = 0$  for all  $j \in \mathbb{N}$ . It then implies that  $\langle \varphi, e_j \rangle = 0$  whenever  $j \in \mathbb{N} \setminus \mathcal{S}$ . Thus  $\ker(S_2) \subset \bar{\mathcal{M}}$ .

On the other hand, if  $j \in \mathbb{N} \setminus \mathcal{S}$  then  $\omega(f_1, j) \cdots \omega(f_{m_1}, j) \neq 0$ , and hence,

$$e_j = \frac{1}{\omega(f_1, j) \cdots \omega(f_{m_1}, j)} T_{f_1} \cdots T_{f_{m_1}} e_j = \frac{1}{\omega(f_1, j) \cdots \omega(f_{m_1}, j)} S_1 e_j.$$

This shows that  $\mathcal{N} \subset S_1(\mathcal{N}) \subset S_1(\mathcal{P})$ . Hence the domain of the operator  $S_2 T_f S_1$  contains  $\mathcal{P}$ , which is dense in  $A^2$ .

Now suppose that  $S_2 T_f S_1(\mathcal{P})$  is of finite dimensions, spanned by the set  $\{u_1, \dots, u_N\}$ . Let  $v_j \in A^2$  such that  $S_2 v_j = u_j$  for  $j = 1, \dots, N$ . It then follows that  $T_f S_1(\mathcal{P})$  is contained in  $\text{Span}(\ker(S_2) \cup \{v_1, \dots, v_N\})$ , which is a subspace of  $\text{Span}(\bar{\mathcal{M}} \cup \{v_1, \dots, v_N\})$ . But as we have seen above,  $\mathcal{N}$  is a subspace of  $S_1(\mathcal{P})$ . So we conclude that  $T_f(\mathcal{N}) \subset \text{Span}(\bar{\mathcal{M}} \cup \{v_1, \dots, v_N\})$ . Corollary 3.2 then implies that  $f$  is the zero function.  $\square$

**Remark 3.4.** *Suppose  $\mathcal{S} \subset \mathbb{N}$  such that  $\sum_{s \in \mathcal{S}} \frac{1}{s+1} < \infty$ . Let  $\mathcal{N}$  (respectively,  $\mathcal{M}$ ) is the subspace of  $\mathcal{P}$  spanned by  $\{e_m : m \in \mathbb{N} \setminus \mathcal{S}\}$  (respectively,  $\{e_m : m \in \mathcal{S}\}$ ). From the proof of Theorem 3.3, we see that if  $S_1, S_2$  are bounded operators on  $A^2$  such that  $\mathcal{N} \subset S_1(\mathcal{P})$ ,  $\ker(S_2) \subset \bar{\mathcal{M}}$  and  $S_2 T_f S_1$  has finite rank then  $f$  must be zero. This shows that the conclusion of Theorem 3.3 remains valid if  $f_j(re^{i\theta}) = \tilde{f}_j(r)e^{is_j\theta}$  and  $g_k(re^{i\theta}) = \tilde{g}_k(r)e^{it_k\theta}$  for bounded functions  $\tilde{f}_j, \tilde{g}_k$  on  $[0, 1)$  and integers  $s_j, t_k$ , for  $1 \leq j \leq m_1, 1 \leq k \leq m_2$ .*



## REFERENCES

- [1] Patrick Ahern and Željko Čučković, *A theorem of Brown-Halmos type for Bergman space Toeplitz operators*, J. Funct. Anal. **187** (2001), no. 1, 200–210. MR MR1867348 (2002h:47040)
- [2] ———, *Some examples related to the Brown-Halmos theorem for the Bergman space*, Acta Sci. Math. (Szeged) **70** (2004), no. 1-2, 373–378. MR MR2072710 (2005d:47046)
- [3] Arlen Brown and Paul R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102. MR MR0160136 (28 #3350)
- [4] Željko Čučković, *Berezin versus Mellin*, J. Math. Anal. Appl. **287** (2003), no. 1, 234–243. MR MR2010267 (2004g:47037)
- [5] Željko Čučković and N. V. Rao, *Mellin transform, monomial symbols, and commuting Toeplitz operators*, J. Funct. Anal. **154** (1998), no. 1, 195–214. MR MR1616532 (99f:47033)
- [6] Kunyu Guo, Shunhua Sun, and Dechao Zheng, *Finite rank commutators and semi-commutators of Toeplitz operators with harmonic symbols*, Illinois J. Math. **51** (2007), no. 2, 583–596 (electronic). MR MR2342676
- [7] Trieu Le, *Diagonal Toeplitz operators on weighted Bergman spaces*, preprint.
- [8] Daniel H. Luecking, *Finite rank Toeplitz operators on the Bergman space*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1717–1723.
- [9] Rubén A. Martínez-Avendaño and Peter Rosenthal, *An introduction to operators on the Hardy-Hilbert space*, Graduate Texts in Mathematics, vol. 237, Springer, New York, 2007. MR MR2270722 (2007k:47051)
- [10] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR MR924157 (88k:00002)

TRIEU LE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO,  
 ONTARIO, CANADA M5S 2E4  
*E-mail address:* `trieu.le@utoronto.ca`